ON THE DEFINITE INTEGRAL OF TWO CONFLUENT HYPERGEOMETRIC FUNCTIONS RELATED TO THE KAMPÉ DE FÉRIET DOUBLE SERIES

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ABSTRACT. We analyze the first-order nonhomogeneous differential equation $-zf'(z)+\beta f(z)=E(\alpha+1;\delta;z)E(\alpha+\gamma;\beta;-z)$ on $\mathbb{C}$, where $E(\cdot)$ denotes the MacRobert's E-function. We show that f can be represented in terms of the double series $F^{1:1;1}_{1:1;1}$, also known as the Kampé de Fériet function. As an application, we demonstrate that the integrals of type $\int_0^\rho t^{\beta+l}E(\cdot;i/t)E(\cdot;-i/t)dt$, where $\rho\in\mathbb{R}$ is arbitrary and $l=0,1,\ldots$, are given in terms of f.
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1. Introduction

In the first part of the present letter we give some basic properties of the solution to the first-order differential equation,

$$\left(-\frac{d}{dz}+\frac{\beta}{z}\right)f(z)=\frac{1}{z}E(\alpha+1;\delta;z)E(\alpha+\gamma;\beta;-z) \tag{1}$$

($z,\alpha,\beta,\gamma,\delta\in\mathbb{C}$ are such that the right-hand side exists), where $E(\cdot)$ denotes the MacRobert's E-function [Mac54, Mac58]

$$E\begin{pmatrix}a_1,a_2,\ldots,a_p\\ b_1,b_2,\ldots,b_q\end{pmatrix};z\Bigg)=\frac{\prod_{j=1}^p\Gamma(a_j)}{\prod_{j=1}^q\Gamma(b_j)}\,{}_pF_q\left(\begin{matrix}a_1,a_2,\ldots,a_p\\ b_1,b_2,\ldots,b_q\end{matrix};-\frac{1}{z}\right)\quad(p\le q) \tag{2a}$$

$$=\frac{1}{2\pi i}\int_B\frac{z^\zeta\Gamma(\zeta)\prod_{j=1}^p\Gamma(a_j-\zeta)}{\prod_{j=1}^q\Gamma(b_j-\zeta)}d\zeta\quad(|\arg z|<\pi) \tag{2b}$$

($\{a_j\in\mathbb{C}\colon\ j=1,\ldots,p\};\{b_j\in\mathbb{C}\colon\ j=1,\ldots,q\};z\in\mathbb{C}$), where $\Gamma(\cdot)$ is the gamma function, ${}_pF_q(\cdot)$ is the generalized hypergeometric function. The Barnes-type contour B is taken up the $\mathrm{Im}\,\zeta$-axis to ensure that the poles of $\{\Gamma(a_j-\zeta)\colon\ j=1,\ldots,p\}$ and $\{\Gamma(b_j-\zeta)\colon\ j=1,\ldots,q\}$ lie to the right

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of the contour and the poles of $\Gamma(\zeta)$ to the left of the contour. In particular, ${}_1F_1(\cdot;\cdot;\cdot) \equiv M(\cdot;\cdot;\cdot)$ where $M(\cdot)$ is the confluent hypergeometric function of the first kind (Kummer's function) [AS72, §6, §13].

We identify $f\colon \mathbb{C} \to \mathbb{C}$ by a convenient notation

$$\begin{pmatrix} \alpha, & \gamma \\ \beta, & \delta \end{pmatrix};z$$

and show that it can be represented by a double series

$$\begin{pmatrix} \alpha, & \gamma \\ \beta, & \delta \end{pmatrix};z = \sum_{\nu=0}^{\infty} \frac{z^{-\nu}\Gamma(\alpha+\gamma+\nu)}{\nu!\Gamma(\beta+\nu)} E\begin{pmatrix} \alpha+1, & \beta+\nu \\ \delta, & \beta+1+\nu \end{pmatrix};z \tag{3a}$$

$$= \frac{1}{2\pi i}\int_B \frac{z^{\zeta}\Gamma(\zeta)\Gamma(\alpha+1-\zeta)}{\Gamma(\delta-\zeta)} E\begin{pmatrix} \alpha+\gamma, & \beta-\zeta \\ \beta, & \beta+1-\zeta \end{pmatrix};-z\, d\zeta, \tag{3b}$$

$$= \frac{\Gamma(\alpha+1)\Gamma(\alpha+\gamma)}{\Gamma(\beta+1)\Gamma(\delta)} F^{1:1;1}_{1:1;1}\left[\begin{matrix} \beta: & \alpha+1; & \alpha+\gamma; \\ \beta+1: & \delta; & \beta; \end{matrix} -\frac{1}{z}, \frac{1}{z}\right], \tag{3c}$$

where the integral representation (3b) is obtained from the series representation by substituting (2b) in (3a) and summing over ν. In (3c), the Kampé de Fériet function $F^{1:1;1}_{1:1;1}$ of two variables is defined in agreement with [GS91],

$$F^{1:1;1}_{1:1;1}\left[\begin{matrix} a_1: & b_1; & b_2; \\ c_1: & d_1; & d_2; \end{matrix} z_1, z_2\right] = \sum_{m_1,m_2=0}^{\infty} \frac{(a_1)_{m_1+m_2}(b_1)_{m_1}(b_2)_{m_2} z_1^{m_1} z_2^{m_2}}{(c_1)_{m_1+m_2}(d_1)_{m_1}(d_2)_{m_2} m_1! m_2!} \tag{4}$$

($(a)_m$ the Pochhammer symbol). If we substitute (2a) in (3a) and then express ${}_2F_2$ as an infinite series, equation (3c) holds by (4).

We also give the asymptotic expansion of f and some recurrence relations.

Although the first-order nonhomogeneous differential equation (1) is of a rather special type, in the second part of the present letter we show, as an illustrative example of possible application of functions f, that

$$\int_0^{\rho} t^{\beta} E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt$$
$$= \rho^{\beta+1}E(\alpha;\beta;i/\rho)E\begin{pmatrix} \alpha+\gamma, & \beta+1 \\ \beta, & \beta+2 \end{pmatrix};-\frac{i}{\rho}$$
$$-i\rho^{\beta+2}\beta\begin{pmatrix} \alpha, & \gamma \\ \beta+2, & \beta+1 \end{pmatrix};\frac{i}{\rho} - \rho^{\beta+3}\begin{pmatrix} \alpha, & \gamma+1 \\ \beta+3, & \beta+1 \end{pmatrix};\frac{i}{\rho} \tag{5}$$

where $\alpha,\beta,\gamma\in\mathbb{C}$; $\operatorname{Re}\beta>-1$; $\rho\in\mathbb{R}$ is arbitrary. In terms of the Kummer's function M, the integrals of type (5) are frequently dealt with, for example, in atomic physics when one considers the normalization of regular Coulomb wave functions [AS72, §14] of the radial Schrödinger equation with angular momentum l and the strength of the Coulomb interaction g. In this case parameters $\alpha=l+1\mp ik$, $\beta=2l+2$ and $\gamma=\pm 2ik$, where $k\sim|g|$ is real and positive. Another equivalent expression of (5) is given in (26).

Note that (5) is divergent as $\rho\to\pm\infty$. Some generalization of (5) is examined as well; see (28). By virtue of (28) we show another proof of the integral (29) that has been studied in [SH03].

Throughout, the parameters $\alpha,\beta,\gamma,\delta\in\mathbb{C}$ and the arguments $z\in\mathbb{C}$ and $\rho,t\in\mathbb{R}$ are assumed to be such that the hypergeometric functions as well as the gamma functions are well-defined unless additional conditions are imposed.

2. Solution

In this paragraph we prove that (3a) solves (1). The equivalence of the remaining two expressions, (3b)–(3c), has already been shown above.

Proceeding from (2a) one deduces that

$$z\frac{d}{dz}E\begin{pmatrix}\alpha+1, & \beta+\nu\\ \delta, & \beta+1+\nu\end{pmatrix};z\Bigg)=(\beta+\nu)E\begin{pmatrix}\alpha+1, & \beta+\nu\\ \delta, & \beta+1+\nu\end{pmatrix};z\Bigg)-E(\alpha+1;\delta;z). \tag{6}$$

Apply zd/dz to (3a), then substitute (6) in obtained expression and get

$$\begin{aligned}
z\frac{d}{dz}\left(\begin{matrix}\alpha, & \gamma\\ \beta, & \delta\end{matrix};z\right)&=-\sum_{\nu=1}^{\infty}\frac{z^{-\nu}\Gamma(\alpha+\gamma+\nu)}{(\nu-1)!\Gamma(\beta+\nu)}E\left(\begin{matrix}\alpha+1, & \beta+\nu\\ \delta, & \beta+1+\nu\end{matrix};z\right)\\
&\quad+\sum_{\nu=0}^{\infty}\frac{z^{-\nu}\Gamma(\alpha+\gamma+\nu)}{\nu!\Gamma(\beta+\nu)}z\frac{d}{dz}E\left(\begin{matrix}\alpha+1, & \beta+\nu\\ \delta, & \beta+1+\nu\end{matrix};z\right)\\
&=-\frac{1}{z}\left(\begin{matrix}\alpha, & \gamma+1\\ \beta+1, & \delta\end{matrix};z\right)+\sum_{\nu=0}^{\infty}\frac{z^{-\nu}\Gamma(\alpha+\gamma+\nu)}{\nu!\Gamma(\beta+\nu)}\\
&\quad\times\left(-E(\alpha+1;\delta;z)+(\beta+\nu)E\left(\begin{matrix}\alpha+1, & \beta+\nu\\ \delta, & \beta+1+\nu\end{matrix};z\right)\right)\\
&=-\frac{1}{z}\left(\begin{matrix}\alpha, & \gamma+1\\ \beta+1, & \delta\end{matrix};z\right)-E(\alpha+1;\delta;z)E(\alpha+\gamma;\beta;-z)\\
&\quad+\beta\left(\begin{matrix}\alpha, & \gamma\\ \beta, & \delta\end{matrix};z\right)+\sum_{\nu=1}^{\infty}\frac{z^{-\nu}\Gamma(\alpha+\gamma+\nu)}{(\nu-1)!\Gamma(\beta+\nu)}
\end{aligned}$$

$$\times E\begin{pmatrix}\alpha+1, & \beta+\nu \\ \delta, & \beta+1+\nu\end{pmatrix};z\Big)$$

$$= -E(\alpha+1;\delta;z)E(\alpha+\gamma;\beta;-z)+\beta\begin{pmatrix}\alpha, & \gamma \\ \beta, & \delta\end{pmatrix};z\Big)$$

as required.

3. Convergence

By expanding (3a) or (3c)–(4) explicitly

$$\frac{\Gamma(\alpha+1)\Gamma(\alpha+\gamma)}{\beta\Gamma(\beta)\Gamma(\delta)}+\left(\frac{\Gamma(\alpha+1)\Gamma(\alpha+\gamma+1)}{\Gamma(\beta+2)\Gamma(\delta)}-\frac{\Gamma(\alpha+2)\Gamma(\alpha+\gamma)}{(\beta+1)\Gamma(\beta)\Gamma(\delta+1)}\right)\cdot\frac{1}{z}$$
$$-\frac{\Gamma(\alpha+2)\Gamma(\alpha+\gamma+1)}{(\beta+2)\Gamma(\beta+1)\Gamma(\delta+1)}\cdot\frac{1}{z^2}+O(z^{-3}) \tag{7}$$

we deduce that the series converges uniformly for $|z|\geq 1$ (the parameters are such that gammas are well-defined).

Indeed, let

$$\begin{pmatrix}\alpha, & \gamma \\ \beta, & \delta\end{pmatrix};z\Big)=\frac{\Gamma(\alpha+1)}{\Gamma(\delta)}\sum_{\nu=0}^{\infty}u_\nu(z), \quad \text{where by (2a) and (3a),}$$

$$u_\nu(z)=\frac{z^{-\nu}\Gamma(\alpha+\gamma+\nu)}{\nu!\Gamma(\beta+1+\nu)}\,{}_2F_2\left(\begin{matrix}\alpha+1, & \beta+\nu \\ \delta, & \beta+1+\nu\end{matrix};-\frac{1}{z}\right). \tag{8}$$

We show that for all $\varepsilon>0$, there exists $N=N(\varepsilon)$ such that, for $\nu>N$ and for $p=1,2,\ldots,$

$$|u_{\nu+1}(z)+u_{\nu+2}(z)+\ldots+u_{\nu+p}(z)|<\varepsilon \quad \text{for all} \quad |z|\geq 1. \tag{9}$$

One can choose N large enough so that for $\nu>N$,

$$\Gamma(\nu)\sim\sqrt{2\pi\nu}(\nu/e)^{\nu}$$

(the Stirling's formula). But

$${}_2F_2\left(\begin{matrix}\alpha+1, & \beta+\nu \\ \delta, & \beta+1+\nu\end{matrix};-\frac{1}{z}\right)\sim M(\alpha+1;\delta;-1/z)$$

for ν large, and so

$$u_\nu(z)\sim\frac{e^{\beta+1-\alpha-\gamma}}{\sqrt{2\pi}}M(\alpha+1;\delta;-1/z)z^{-\nu}e^{\nu}\nu^{\alpha+\gamma-\beta-\nu-3/2}. \tag{10}$$

As seen, $|u_\nu(z)| \le |u_\nu(1)|$ and $u_\nu(z) \to 0$ as $\nu \to \infty$ for $|z| \ge 1$. By applying the Minkowski's inequality to (9) and replacing ν by $\nu + s$ in (10) for $s = 1, 2, \dots, p$, yields

$$\begin{aligned}
\left|\sum_{s=1}^{p} u_{\nu+s}(z)\right| &\le \sum_{s=1}^{p} |u_{\nu+s}(z)| \le \sum_{s=1}^{p} |u_{\nu+s}(1)| \\
&\sim \frac{1}{\sqrt{2\pi}} \left|(e/\nu)^{\nu+\beta+1-\alpha-\gamma} \nu^{-1/2} M(\alpha+1;\delta;-1)\right| \sum_{s=1}^{p} (e/\nu)^s \\
&\sim \frac{e}{\nu!} \left|(e/\nu)^{\beta+1-\alpha-\gamma} M(\alpha+1;\delta;-1)\right| < \varepsilon \quad (\nu \text{ large and } |z| \ge 1)
\end{aligned}$$

for all $p = 1, 2, \dots$, where $\varepsilon > 0$ can be chosen arbitrarily small. This proves that the series (3a) converges uniformly for $|z| \ge 1$.

Since ${}_2F_2(\cdot)$ is entire, that is, its radius of convergence is ∞, one deduces that the series is convergent for $0 < |z| < 1$. We note that all other regions of convergence due to the parameters $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ can be obtained by applying the properties of ${}_2F_2(\cdot)$ to (3a).

4. Asymptotic expansion

By (7),

$$\begin{pmatrix} \alpha, & \gamma \\ \beta, & \delta \end{pmatrix} ; z = \frac{\Gamma(\alpha+1)\Gamma(\alpha+\gamma)}{\Gamma(\beta+1)\Gamma(\delta)} + O(z^{-1}) \quad \text{as} \quad |z| \to \infty. \tag{11}$$

We also note that the analysis of (1) yields the same result. Indeed, functions $M(\cdot;\cdot;\pm 1/z) \to 1$ as $|z| \to \infty$, and so by (2a) the right-hand side of (1) becomes $[\Gamma(\alpha+1)\Gamma(\alpha+\gamma)/(\Gamma(\beta)\Gamma(\delta))] z^{-1}$ up to $O(z^{-2})$. By solving the modified differential equation one obtains (11).

By Paris [Par05],

$${}_2F_2\left(\begin{matrix} \alpha+1, & \beta+\nu \\ \delta, & \beta+1+\nu \end{matrix}; -\frac{1}{z}\right) \sim (-1/z)^{\alpha-\delta} e^{-1/z} \frac{(\beta+\nu)\Gamma(\delta)}{\Gamma(\alpha+1)} \text{ as } |z| \to 0 \tag{12}$$

for $|\arg z| < \pi/2$ and $\operatorname{Re} z < 0$. Substitute (12) in (8) and get

$$\begin{aligned}
\begin{pmatrix} \alpha, & \gamma \\ \beta, & \delta \end{pmatrix} ; z &\sim (-1/z)^{\alpha-\delta} e^{-1/z} \sum_{\nu=0}^{\infty} \frac{z^{-\nu}\Gamma(\alpha+\gamma+\nu)}{\nu!\Gamma(\beta+\nu)} \\
&= \frac{\Gamma(\alpha+\gamma)}{\Gamma(\beta)} (-1/z)^{\alpha-\delta} e^{-1/z} M(\alpha+\gamma;\beta;1/z).
\end{aligned}$$

But [AS72, §13] $M(\alpha+\gamma;\beta;1/z) \sim (-z)^{\alpha+\gamma}\Gamma(\beta)/\Gamma(\beta-\alpha-\gamma)$ as $|z| \to 0$ for $\operatorname{Re} z < 0$, and so

$$\begin{pmatrix}\alpha, & \gamma \\ \beta, & \delta\end{pmatrix}; z \sim e^{-1/z}(-z)^{\gamma+\delta}\frac{\Gamma(\alpha+\gamma)}{\Gamma(\beta-\alpha-\gamma)} \tag{13}$$

as $|z| \to 0$. Note that (13) as well as the asymptotic expansion of f for $\operatorname{Re} z > 0$ can be established by a direct analysis of (1). In the latter case the right-hand side of (1) is modified to $e^{1/z}z^{\beta-\gamma}\Gamma(\alpha+1)/\Gamma(\delta-\alpha-1)$ ($|z| \to 0; \operatorname{Re} z > 0$), and thus

$$\begin{pmatrix}\alpha, & \gamma \\ \beta, & \delta\end{pmatrix}; z \sim e^{1/z}z^{\beta+2-\gamma}\frac{\Gamma(\alpha+1)}{\Gamma(\delta-\alpha-1)}$$

as $|z| \to 0$ ($\operatorname{Re} z > 0$).

5. Some recurrence relations

By using the recurrence relations for E-functions [in [Bho62] substitute $p = q = 2$, $\alpha_1 = \alpha+1$, $\alpha_2 = \beta+\nu$, $\rho_1 = \delta$, $\rho_2 = \beta+1+\nu$ and get (14a); in [Erd53, §5.2.1, eqs. (7) and (9)] substitute $\alpha_1 = \alpha+1$, $\alpha_2 = \beta+\nu$, $\rho_1 = \delta$, $\rho_2 = \beta+1+\nu$ and get (14b)–(14c)] one obtains from (3a) that

$$E(\alpha+1;\delta;z)E(\alpha+\gamma;\beta+1;-z) = \begin{pmatrix}\alpha, & \gamma \\ \beta, & \delta\end{pmatrix}; z - \frac{1}{z}\begin{pmatrix}\alpha+1, & \gamma-1 \\ \beta+1, & \delta+1\end{pmatrix}; z \tag{14a}$$

$$= \frac{\beta-\alpha-1}{\beta}\begin{pmatrix}\alpha, & \gamma \\ \beta, & \delta\end{pmatrix}; z + \frac{1}{\beta}\begin{pmatrix}\alpha+1, & \gamma-1 \\ \beta, & \delta\end{pmatrix}; z + \frac{1}{\beta z^2}\begin{pmatrix}\alpha+1, & \gamma \\ \beta+2, & \delta+1\end{pmatrix}; z \tag{14b}$$

$$= \frac{\beta-\delta+1}{\beta}\begin{pmatrix}\alpha, & \gamma \\ \beta, & \delta\end{pmatrix}; z + \frac{1}{\beta}\begin{pmatrix}\alpha, & \gamma \\ \beta, & \delta-1\end{pmatrix}; z + \frac{1}{\beta z^2}\begin{pmatrix}\alpha+1, & \gamma \\ \beta+2, & \delta+1\end{pmatrix}; z. \tag{14c}$$

Since the proofs are obvious due to the properties of E-functions we will omit them by noting that the recurrence relations (14b)–(14c) are reduced by applying (14a) so that the sum

$$\sum_{\nu=0}^{\infty}\frac{z^{-\nu}\Gamma(\alpha+\gamma+\nu)}{\nu!\Gamma(\beta+\nu)}E\begin{pmatrix}\alpha+2, & \beta+1+\nu \\ \delta+1, & \beta+2+\nu\end{pmatrix}; z$$

can be calculated by setting $\Gamma(\beta+\nu) = \Gamma(\beta+1+\nu)/(\beta+\nu)$, thus yielding

$$\beta\begin{pmatrix}\alpha+1, & \gamma-1 \\ \beta+1, & \delta+1\end{pmatrix}; z + \frac{1}{z}\begin{pmatrix}\alpha+1, & \gamma \\ \beta+2, & \delta+1\end{pmatrix}; z$$

where the first summand is then represented by (14a).

6. Some series for E-function

Here we prove the sums arising in the subsequent paragraphs:

$$\sum_{s=1}^{\infty}\frac{(-1)^s}{(\beta+1)^s}E\left(\begin{matrix}\alpha+1, & [2]_{s-1}\\ \beta+1, & [1]_{s-1}\end{matrix};z\right)=-E\left(\begin{matrix}\alpha+1, & \beta+2\\ \beta+1, & \beta+3\end{matrix};z\right), \tag{15a}$$

$$\sum_{s=1}^{\infty}\frac{(-1)^s(r+s)!}{s!(\beta+1)^s}E\left(\begin{matrix}\alpha+1, & [2]_{s-1}\\ \beta+1, & [1]_{s-1}\end{matrix};z\right)=zr!\Bigg(E(\alpha;\beta;z)$$
$$-(\beta+1)^{r+1}E\left(\begin{matrix}\alpha, & [\beta+1]_{r+1}\\ \beta, & [\beta+2]_{r+1}\end{matrix};z\right)\Bigg)\quad (r=0,1,\ldots), \tag{15b}$$

and

$$\sum_{s=1}^{\infty}(-1)^sE\left(\begin{matrix}\alpha+1, & [2]_{s-1}\\ \beta+1, & [1]_{s-1}\end{matrix};z\right)E\left(\begin{matrix}\alpha+\gamma, & [\beta+1]_{s+1}\\ \beta, & [\beta+2]_{s+1}\end{matrix};-z\right)$$
$$=-\beta\left(\begin{matrix}\alpha, & \gamma\\ \beta+2, & \beta+1\end{matrix};z\right)-\frac{1}{z}\left(\begin{matrix}\alpha, & \gamma+1\\ \beta+3, & \beta+1\end{matrix};z\right), \tag{15c}$$

where $[a]_n$ ($a\in\mathbb{C};n=0,1,\ldots$) denotes, for short, the set of parameters $a,a,\ldots,a$ (n times).

Apply (2b) to the left-hand side of (15a) and write

$$\sum_{s=1}^{\infty}\frac{(-1)^s}{(\beta+1)^s}E\left(\begin{matrix}\alpha+1, & [2]_{s-1}\\ \beta+1, & [1]_{s-1}\end{matrix};z\right)=\frac{1}{2\pi i}\int_B d\zeta\,\frac{z^{\zeta}\Gamma(\zeta)\Gamma(\alpha+1-\zeta)}{\Gamma(\beta+1-\zeta)}$$
$$\times\sum_{s=1}^{\infty}\frac{(-1)^s(1-\zeta)^{s-1}}{(\beta+1)^s}=-\frac{1}{2\pi i}\int_B d\zeta\,\frac{z^{\zeta}\Gamma(\zeta)\Gamma(\alpha+1-\zeta)}{\Gamma(\beta+1-\zeta)}\cdot\frac{1}{\beta+2-\zeta}.$$

By applying (2b) along with $(\beta+2-\zeta)=\Gamma(\beta+3-\zeta)/\Gamma(\beta+2-\zeta)$ to the latter contour integral yields (15a).

Similarly, apply (2b) to the left-hand side of (15b) and get

$$\sum_{s=1}^{\infty}\frac{(-1)^s(r+s)!}{s!(\beta+1)^s}E\left(\begin{matrix}\alpha+1, & [2]_{s-1}\\ \beta+1, & [1]_{s-1}\end{matrix};z\right)=\frac{1}{2\pi i}\int_B d\zeta\,\frac{z^{\zeta}\Gamma(\zeta)\Gamma(\alpha+1-\zeta)}{\Gamma(\beta+1-\zeta)}$$
$$\times\sum_{s=1}^{\infty}\frac{(-1)^s(r+s)!(1-\zeta)^{s-1}}{s!(\beta+1)^s}=\frac{1}{2\pi i}\int_B d\zeta\,\frac{z^{\zeta}\Gamma(\zeta)\Gamma(\alpha+1-\zeta)}{\Gamma(\beta+1-\zeta)}\cdot\frac{r!}{1-\zeta}$$
$$\times\left(-1+\frac{(\beta+1)^{r+1}}{(\beta+2-\zeta)^{r+1}}\right)=-\frac{r!}{2\pi i}\int_B d\zeta\,\frac{z^{\zeta}\Gamma(\zeta)\Gamma(\alpha+1-\zeta)}{\Gamma(\beta+1-\zeta)(1-\zeta)}+\frac{r!(\beta+1)^{r+1}}{2\pi i}$$
$$\times\int_B d\zeta\,\frac{z^{\zeta}\Gamma(\zeta)\Gamma(\alpha+1-\zeta)}{\Gamma(\beta+1-\zeta)(1-\zeta)(\beta+2-\zeta)^{r+1}}. \tag{16}$$

In order to calculate the obtained contour integrals, two possibilities are valid: either apply (2b) and then reduce the order of obtained E-functions or make use of the residues directly recalling that $\mathrm{Res}_{\zeta=-\nu}\Gamma(\zeta)=(-1)^\nu/\nu!$ ($\nu=0,1,\ldots$). The combination of both strategies yields

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_B d\zeta\,\frac{z^\zeta\Gamma(\zeta)\Gamma(\alpha+1-\zeta)}{\Gamma(\beta+1-\zeta)(1-\zeta)}=E\begin{pmatrix}\alpha+1, & 1\\ \beta+1, & 2\end{pmatrix};z\Bigg)=z\left(\frac{\Gamma(\alpha)}{\Gamma(\beta)}-E(\alpha;\beta;z)\right),\\
&\frac{1}{2\pi i}\int_B d\zeta\,\frac{z^\zeta\Gamma(\zeta)\Gamma(\alpha+1-\zeta)}{\Gamma(\beta+1-\zeta)(1-\zeta)(\beta+2-\zeta)^{r+1}}=E\left(\begin{matrix}\alpha+1, & [\beta+2]_{r+1}, & 1\\ \beta+1, & [\beta+3]_{r+1}, & 2\end{matrix};z\right)\\
&=\sum_{\nu=0}^{\infty}\frac{(-1/z)^\nu\Gamma(\alpha+1+\nu)}{(\nu+1)!\Gamma(\beta+1+\nu)(\beta+2+\nu)^{r+1}}=\frac{z\Gamma(\alpha)}{\Gamma(\beta)(\beta+1)^{r+1}}\\
&+\sum_{\nu=0}^{\infty}\frac{(-1/z)^{\nu-1}\Gamma(\alpha+\nu)}{\nu!\Gamma(\beta+\nu)(\beta+1+\nu)^{r+1}}=\frac{z\Gamma(\alpha)}{\Gamma(\beta)(\beta+1)^{r+1}}-zE\left(\begin{matrix}\alpha, & [\beta+1]_{r+1}\\ \beta, & [\beta+2]_{r+1}\end{matrix};z\right).
\end{aligned}
$$

Substitute obtained expressions in (16) and get (15b).

The application of (2b) to the left-hand side of (15c) gives the integrals

$$
\begin{aligned}
&\sum_{s=1}^{\infty}(-1)^s E\left(\begin{matrix}\alpha+1, & [2]_{s-1}\\ \beta+1, & [1]_{s-1}\end{matrix};z\right)E\left(\begin{matrix}\alpha+\gamma, & [\beta+1]_{s+1}\\ \beta, & [\beta+2]_{s+1}\end{matrix};-z\right)\\
&=\frac{1}{2\pi i}\int_B d\zeta\,\frac{z^\zeta\Gamma(\zeta)\Gamma(\alpha+1-\zeta)}{\Gamma(\beta+1-\zeta)}\cdot\frac{1}{2\pi i}\int_B d\zeta'\,\frac{(-z)^{\zeta'}\Gamma(\zeta')\Gamma(\alpha+\gamma-\zeta')}{\Gamma(\beta-\zeta')}\\
&\times\sum_{s=1}^{\infty}\frac{(-1)^s(1-\zeta)^{s-1}}{(\beta+1-\zeta')^{s+1}}=-\frac{1}{2\pi i}\int_B d\zeta\,\frac{z^\zeta\Gamma(\zeta)\Gamma(\alpha+1-\zeta)}{\Gamma(\beta+1-\zeta)}\\
&\times\frac{1}{2\pi i}\int_B d\zeta'\,\frac{(-z)^{\zeta'}\Gamma(\zeta')\Gamma(\alpha+\gamma-\zeta')}{\Gamma(\beta-\zeta')(\beta+1-\zeta')(\beta+2-\zeta-\zeta')}\\
&=-\frac{1}{2\pi i}\int_B d\zeta\,\frac{z^\zeta\Gamma(\zeta)\Gamma(\alpha+1-\zeta)}{\Gamma(\beta+1-\zeta)}E\left(\begin{matrix}\alpha+\gamma, & \beta+1, & \beta+2-\zeta\\ \beta, & \beta+2, & \beta+3-\zeta\end{matrix};-z\right).
\end{aligned}
$$

By the recurrence relation for E-function [in [Erd53, §5.2.1, eq. (9)] substitute $\alpha_1=\alpha+\gamma$, $\alpha_2=\beta+1$, $\alpha_3=\beta+2-\zeta$, $\rho_1=\beta+1$, $\rho_2=\beta+2$, $\rho_3=\beta+3-\zeta$],

$$
\begin{aligned}
&E\left(\begin{matrix}\alpha+\gamma, & \beta+1, & \beta+2-\zeta\\ \beta, & \beta+2, & \beta+3-\zeta\end{matrix};-z\right)=\beta E\left(\begin{matrix}\alpha+\gamma, & \beta+2-\zeta\\ \beta+2, & \beta+3-\zeta\end{matrix};-z\right)\\
&+\frac{1}{z}E\left(\begin{matrix}\alpha+\gamma+1, & \beta+3-\zeta\\ \beta+3, & \beta+4-\zeta\end{matrix};-z\right)
\end{aligned}
$$

and so

$$
\begin{aligned}
&\sum_{s=1}^{\infty}(-1)^s E\begin{pmatrix}\alpha+1, & [2]_{s-1} \\ \beta+1, & [1]_{s-1}\end{pmatrix}; z\Bigg) E\begin{pmatrix}\alpha+\gamma, & [\beta+1]_{s+1} \\ \beta, & [\beta+2]_{s+1}\end{pmatrix}; -z\Bigg) \\
&= -\frac{\beta}{2\pi i}\int_B d\zeta \, \frac{z^\zeta \Gamma(\zeta)\Gamma(\alpha+1-\zeta)}{\Gamma(\beta+1-\zeta)} E\begin{pmatrix}\alpha+\gamma, & \beta+2-\zeta \\ \beta+2, & \beta+3-\zeta\end{pmatrix}; -z\Bigg) \\
&- \frac{1}{2\pi i z}\int_B d\zeta \, \frac{z^\zeta \Gamma(\zeta)\Gamma(\alpha+1-\zeta)}{\Gamma(\beta+1-\zeta)} E\begin{pmatrix}\alpha+\gamma+1, & \beta+3-\zeta \\ \beta+3, & \beta+4-\zeta\end{pmatrix}; -z\Bigg).
\end{aligned}
$$

But these two integrals are nothing more than the integral representations of function f, where the parameters β, γ, δ in (3b) are accordingly replaced by $\beta+2$, γ, $\beta+1$ in the first integral and by $\beta+3$, $\gamma+1$, $\beta+1$ in the second one. Therefore, equation (15c) holds.

7. Definite integral

In order to calculate the integral (5) we make use of the series expansion (2a) of E-functions (equivalently, apply (3c)–(4)),

$$
\int_0^\rho t^\beta E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt = \sum_{\mu,\nu=0}^{\infty} \frac{i^\mu(-i)^\nu \Gamma(\alpha+\mu)\Gamma(\alpha+\gamma+\nu)}{\mu!\nu!\Gamma(\beta+\mu)\Gamma(\beta+\nu)} \times \int_0^\rho t^{\beta+\mu+\nu} dt. \tag{17}
$$

The integral on the right-hand side equals $\rho^{\beta+\mu+\nu+1}/(\beta+\mu+\nu+1)$ provided $\operatorname{Re}\beta > -1$. Expand $(\beta+\mu+\nu+1)^{-1}$ into the binomial series,

$$
\frac{1}{\beta+\mu+\nu+1} = \frac{1}{\beta+1}\sum_{r,s=0}^{\infty} \frac{(-1)^{r+s}\mu^r \nu^s (r+s)!}{r!s!(\beta+1)^{r+s}} \tag{18}
$$

and get

$$
\int_0^\rho t^\beta E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt = \frac{\rho^{\beta+1}}{\beta+1}\sum_{r,s=0}^{\infty} \frac{(-1)^{r+s}(r+s)!}{r!s!(\beta+1)^{r+s}} \times \sum_{\mu=0}^{\infty} \frac{(i\rho)^\mu \mu^r \Gamma(\alpha+\mu)}{\mu!\Gamma(\beta+\mu)} \sum_{\nu=0}^{\infty} \frac{(-i\rho)^\nu \nu^s \Gamma(\alpha+\gamma+\nu)}{\nu!\Gamma(\beta+\nu)}. \tag{19}
$$

In (19) the latter two sums are of the type

$$
\sum_{\nu=0}^{\infty} \frac{z^\nu \nu^r \Gamma(\alpha+\nu)}{\nu!\Gamma(\beta+\nu)} \quad (r = 0,1,\ldots).
$$

For $r = 0$, this is a usual hypergeometric series given by $E(\alpha;\beta;-1/z)$. For $r = 1$, make a substitution $\nu \to \nu+1$ and get $zE(\alpha+1;\beta+1;-1/z)$. For $r \geq 2$, write

$$\sum_{\nu=0}^{\infty} \frac{z^\nu \nu^r \Gamma(\alpha+\nu)}{\nu!\Gamma(\beta+\nu)} = z\sum_{\nu=0}^{\infty} \frac{z^\nu (1+\nu)^{r-1}\Gamma(\alpha+1+\nu)}{\nu!\Gamma(\beta+1+\nu)}$$
$$= z\sum_{\nu=0}^{\infty} \frac{z^\nu \Gamma(\alpha+1+\nu)[\Gamma(2+\nu)]^{r-1}}{\nu!\Gamma(\beta+1+\nu)[\Gamma(1+\nu)]^{r-1}} = zE\begin{pmatrix}\alpha+1, & [2]_{r-1}; -\dfrac{1}{z}\\ \beta+1, & [1]_{r-1}\end{pmatrix}. \tag{20}$$

Note that (20) also holds for $r = 1$, provided $[a]_0$ is the empty set.

In (19) expand both sums over $r \geq 0$ and $s \geq 0$ into two terms, one with $r = 0$ and $s = 0$ and another with sums over $r \geq 1$ and $s \geq 1$. Then substitute the right-hand side of (20) in obtained expression for $r \geq 1$ and $s \geq 1$, and $E(\alpha;\beta;i/\rho)$ and $E(\alpha+\gamma;\beta;-i/\rho)$ accordingly for $r = 0$ and $s = 0$. The result reads

$$\int_0^\rho t^\beta E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt = \frac{\rho^{\beta+1}}{\beta+1}\Bigg(E(\alpha;\beta;i/\rho)E(\alpha+\gamma;\beta;-i/\rho)$$
$$-i\rho E(\alpha;\beta;i/\rho)\sum_{s=1}^{\infty}\frac{(-1)^s}{(\beta+1)^s}E\begin{pmatrix}\alpha+\gamma+1, & [2]_{s-1}; -\dfrac{i}{\rho}\\ \beta+1, & [1]_{s-1}\end{pmatrix} + i\rho E(\alpha+\gamma;\beta;-i/\rho)$$
$$\times\sum_{s=1}^{\infty}\frac{(-1)^s}{(\beta+1)^s}E\begin{pmatrix}\alpha+1, & [2]_{s-1}; \dfrac{i}{\rho}\\ \beta+1, & [1]_{s-1}\end{pmatrix} + \rho^2\sum_{r=1}^{\infty}\frac{(-1)^r}{r!(\beta+1)^r}E\begin{pmatrix}\alpha+1, & [2]_{r-1}; \dfrac{i}{\rho}\\ \beta+1, & [1]_{r-1}\end{pmatrix}$$
$$\times\sum_{s=1}^{\infty}\frac{(-1)^s(r+s)!}{s!(\beta+1)^s}E\begin{pmatrix}\alpha+\gamma+1, & [2]_{s-1}; -\dfrac{i}{\rho}\\ \beta+1, & [1]_{s-1}\end{pmatrix}\Bigg).$$

The sums over $s \geq 1$ are calculated in (15a)–(15b). By substituting them in the above expression, after some elementary simplifications one obtains the integral

$$\int_0^\rho t^\beta E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt = \frac{\rho^{\beta+1}}{\beta+1}\Bigg[E(\alpha;\beta;i/\rho)\Bigg(E(\alpha+\gamma;\beta;-i/\rho)$$
$$+i\rho E\begin{pmatrix}\alpha+\gamma+1, & \beta+2; -\dfrac{i}{\rho}\\ \beta+1, & \beta+3\end{pmatrix}\Bigg) + i\rho(\beta+1)\sum_{s=1}^{\infty}(-1)^s E\begin{pmatrix}\alpha+1, & [2]_{s-1}; \dfrac{i}{\rho}\\ \beta+1, & [1]_{s-1}\end{pmatrix}$$
$$\times E\begin{pmatrix}\alpha+\gamma, & [\beta+1]_{s+1}; -\dfrac{i}{\rho}\\ \beta, & [\beta+2]_{s+1}\end{pmatrix}\Bigg].$$

The latter sum is given in (15c). By substituting the right-hand side of (15c) in the above expression and noting that [in [Bho62, eq. (4.4)] substitute $p = q = 2$, $\alpha_1 = \alpha+\gamma$, $\alpha_2 = \beta+1$, $\rho_1 = \beta$, $\rho_2 = \beta+2$, and after that apply [Erd53, §5.2.1, eq. (7)] with $\alpha_1 = \alpha+\gamma-1$, $\rho_1 = \beta-1$]

$$E(\alpha+\gamma;\beta;-i/\rho)+i\rho E\begin{pmatrix}\alpha+\gamma+1, & \beta+2 \\ \beta+1, & \beta+3\end{pmatrix}; -\frac{i}{\rho}\Bigg)$$
$$=(\beta+1)E\begin{pmatrix}\alpha+\gamma, & \beta+1 \\ \beta, & \beta+2\end{pmatrix}; -\frac{i}{\rho}\Bigg), \tag{21}$$

one derives (5), as required.

8. Some generalization

The procedure to calculate the integral (5) suggests a natural generalization,

$$\int_0^\rho t^{\beta+l}E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt \tag{22}$$

for some $l=0,1,\ldots$ In this case, the integral on the right-hand side of (17) is calculated by replacing β by $\beta+l$, for $\operatorname{Re}\beta+l>-1$. The subsequent denominator in (18) is replaced by

$$\frac{1}{\beta+l+\mu+\nu+1}=\frac{1}{\beta+1}\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}\nu^s(r+s)!}{r!s!(\beta+1)^{r+s}}\sum_{u=0}^{r}\binom{r}{u}\mu^u l^{r-u},$$

and so (19) becomes equal to

$$\int_0^\rho t^{\beta+l}E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt=\frac{\rho^{\beta+l+1}}{\beta+1}\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}(r+s)!}{r!s!(\beta+1)^{r+s}}$$
$$\times\left(\sum_{u=0}^{r}\binom{r}{u}l^{r-u}\sum_{\mu=0}^{\infty}\frac{(i\rho)^\mu\mu^u\Gamma(\alpha+\mu)}{\mu!\Gamma(\beta+\mu)}\right)\sum_{\nu=0}^{\infty}\frac{(-i\rho)^\nu\nu^s\Gamma(\alpha+\gamma+\nu)}{\nu!\Gamma(\beta+\nu)}. \tag{23}$$

For $s=0$, the sum over ν equals $E(\alpha+\gamma;\beta;-i/\rho)$, whereas for $s=1,2,\ldots$, it is given by (20) with α and z replaced by $\alpha+\gamma$ and $-i\rho$, respectively ($r\equiv s$).

For $r=0$, the sum in the parentheses in (23) equals $E(\alpha;\beta;i/\rho)$, and for $r=1,2,\ldots$, it is given by (after the summation over μ by (20))

$$l^rE(\alpha;\beta;i/\rho)+i\rho\sum_{u=1}^{r}\binom{r}{u}l^{r-u}E\begin{pmatrix}\alpha+1, & [2]_{u-1} \\ \beta+1, & [1]_{u-1}\end{pmatrix}; \frac{i}{\rho}\Bigg)=l^rE(\alpha;\beta;i/\rho)$$
$$+i\rho\sum_{\nu=0}^{\infty}\frac{(i\rho)^\nu\Gamma(\alpha+1+\nu)}{(\nu+1)!\Gamma(\beta+1+\nu)}\sum_{u=1}^{r}\binom{r}{u}(\nu+1)^u l^{r-u}=l^rE(\alpha;\beta;i/\rho)$$
$$-i\rho l^r\sum_{\nu=0}^{\infty}\frac{(i\rho)^\nu\Gamma(\alpha+1+\nu)}{(\nu+1)!\Gamma(\beta+1+\nu)}+i\rho\sum_{\nu=0}^{\infty}\frac{(i\rho)^\nu\Gamma(\alpha+1+\nu)(l+1+\nu)^r}{(\nu+1)!\Gamma(\beta+1+\nu)}$$

$$= l^r E(\alpha;\beta;i/\rho) + l^r\left(\frac{\Gamma(\alpha)}{\Gamma(\beta)} - E(\alpha;\beta;i/\rho)\right) + \sum_{\nu=1}^{\infty}\frac{(i\rho)^{\nu}\Gamma(\alpha+\nu)(l+\nu)^r}{\nu!\Gamma(\beta+\nu)}$$
$$= E\begin{pmatrix}\alpha, & [l+1]_r; & \dfrac{i}{\rho}\\ \beta, & [l]_r & \end{pmatrix}.$$

Combining the results all together and substituting them in (23) we find that

$$\int_0^{\rho} t^{\beta+l}E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt = \frac{\rho^{\beta+l+1}}{\beta+1}\Bigg[E(\alpha;\beta;i/\rho)$$
$$\times E(\alpha+\gamma;\beta;-i/\rho) - i\rho E(\alpha;\beta;i/\rho)\sum_{s=1}^{\infty}\frac{(-1)^s}{(\beta+1)^s}E\begin{pmatrix}\alpha+\gamma+1, & [2]_{s-1}; & -\dfrac{i}{\rho}\\ \beta+1, & [1]_{s-1} & \end{pmatrix}$$
$$+ E(\alpha+\gamma;\beta;-i/\rho)\sum_{s=1}^{\infty}\frac{(-1)^s}{(\beta+1)^s}E\begin{pmatrix}\alpha, & [l+1]_s; & \dfrac{i}{\rho}\\ \beta, & [l]_s & \end{pmatrix} - i\rho\sum_{r=1}^{\infty}\frac{(-1)^r}{r!(\beta+1)^r}$$
$$\times E\begin{pmatrix}\alpha, & [l+1]_r; & \dfrac{i}{\rho}\\ \beta, & [l]_r & \end{pmatrix}\sum_{s=1}^{\infty}\frac{(-1)^s(r+s)!}{s!(\beta+1)^s}E\begin{pmatrix}\alpha+\gamma+1, & [2]_{s-1}; & -\dfrac{i}{\rho}\\ \beta+1, & [1]_{s-1} & \end{pmatrix}\Bigg].$$

Substitute the sums (15a)–(15b) in the above expression and get by applying (21),

$$\int_0^{\rho} t^{\beta+l}E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt$$
$$= \rho^{\beta+l+1}\Bigg[E(\alpha;\beta;i/\rho)E\begin{pmatrix}\alpha+\gamma, & \beta+1; & -\dfrac{i}{\rho}\\ \beta, & \beta+2 & \end{pmatrix}$$
$$+ \sum_{s=1}^{\infty}(-1)^s E\begin{pmatrix}\alpha, & [l+1]_s; & \dfrac{i}{\rho}\\ \beta, & [l]_s & \end{pmatrix}E\begin{pmatrix}\alpha+\gamma, & [\beta+1]_{s+1}; & -\dfrac{i}{\rho}\\ \beta, & [\beta+2]_{s+1} & \end{pmatrix}\Bigg]. \tag{24a}$$

By (2b), the latter sum can be represented by

$$\sum_{s=1}^{\infty}(-1)^s E\begin{pmatrix}\alpha, & [l+1]_s; & \dfrac{i}{\rho}\\ \beta, & [l]_s & \end{pmatrix}E\begin{pmatrix}\alpha+\gamma, & [\beta+1]_{s+1}; & -\dfrac{i}{\rho}\\ \beta, & [\beta+2]_{s+1} & \end{pmatrix}$$
$$= \frac{1}{2\pi i}\int_B d\zeta\,\frac{(i/\rho)^{\zeta}\Gamma(\zeta)\Gamma(\alpha-\zeta)}{\Gamma(\beta-\zeta)}\cdot\frac{1}{2\pi i}\int_B d\zeta'\,\frac{(-i/\rho)^{\zeta'}\Gamma(\zeta')\Gamma(\alpha+\gamma-\zeta')}{\Gamma(\beta-\zeta')}$$
$$\times\sum_{s=1}^{\infty}\frac{(-1)^s(l-\zeta)^s}{(\beta+1-\zeta')^{s+1}} = \frac{1}{2\pi i}\int_B d\zeta\,\frac{(i/\rho)^{\zeta}\Gamma(\zeta)\Gamma(\alpha-\zeta)}{\Gamma(\beta-\zeta)}\cdot\frac{1}{2\pi i}\int_B d\zeta'$$
$$\times\frac{(-i/\rho)^{\zeta'}\Gamma(\zeta')\Gamma(\alpha+\gamma-\zeta')}{\Gamma(\beta-\zeta')}\left(-\frac{1}{\beta+1-\zeta'}+\frac{1}{\beta+l+1-\zeta-\zeta'}\right)$$

$$= -E(\alpha;\beta;i/\rho)E\begin{pmatrix}\alpha+\gamma, & \beta+1 \\ \beta, & \beta+2\end{pmatrix}; -\frac{i}{\rho}\Bigg) + \frac{1}{2\pi i}\int_B d\zeta\, \frac{(i/\rho)^{\zeta}\Gamma(\zeta)\Gamma(\alpha-\zeta)}{\Gamma(\beta-\zeta)}$$
$$\times E\left(\begin{matrix}\alpha+\gamma, & \beta+l+1-\zeta \\ \beta, & \beta+l+2-\zeta\end{matrix}; -\frac{i}{\rho}\right),$$

and so (24a) becomes equal to

$$\int_0^{\rho} t^{\beta+l}E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt$$
$$= \frac{\rho^{\beta+l+1}}{2\pi i}\int_B d\zeta\, \frac{(i/\rho)^{\zeta}\Gamma(\zeta)\Gamma(\alpha-\zeta)}{\Gamma(\beta-\zeta)}E\left(\begin{matrix}\alpha+\gamma, & \beta+l+1-\zeta \\ \beta, & \beta+l+2-\zeta\end{matrix}; -\frac{i}{\rho}\right). \quad (24b)$$

The integral representation (24b) of (22) is more convenient than (24a) because in the latter case one can directly apply the recurrence relation [in [Erd53, §5.2.1, eq. (9)] substitute $\alpha_1 = \alpha+\gamma$, $\alpha_2 = \beta+l+1-\zeta$, $\rho_1 = \beta+1$, $\rho_2 = \beta+l+2-\zeta$ and $x = -i/\rho$]

$$E\left(\begin{matrix}\alpha+\gamma, & \beta+l+1-\zeta \\ \beta, & \beta+l+2-\zeta\end{matrix}; -\frac{i}{\rho}\right) = \beta E\left(\begin{matrix}\alpha+\gamma, & \beta+l+1-\zeta \\ \beta+1, & \beta+l+2-\zeta\end{matrix}; -\frac{i}{\rho}\right)$$
$$- i\rho E\left(\begin{matrix}\alpha+\gamma+1, & \beta+l+2-\zeta \\ \beta+2, & \beta+l+3-\zeta\end{matrix}; -\frac{i}{\rho}\right) \quad (25)$$

$l+1$ times for a given $l = 0,1,\dots$, and then apply (3b) to (24b).

For example, if $l = 0$, then (24b) coincides with (5). Namely, substitute $l = 0$ in (25), and the obtained expression in (24b), and get by applying (3b),

$$\int_0^{\rho} t^{\beta}E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt$$
$$= \rho^{\beta+1}\left(\beta\left(\begin{matrix}\alpha-1, & \gamma+1 \\ \beta+1, & \beta\end{matrix}; \frac{i}{\rho}\right) - i\rho\left(\begin{matrix}\alpha-1, & \gamma+2 \\ \beta+2, & \beta\end{matrix}; \frac{i}{\rho}\right)\right). \quad (26)$$

Apply (14a) to both f functions in the above expression and get (5).

As an example, we also calculate the integral (22) for $l = 1$. In this case apply (25) two times,

$$E\left(\begin{matrix}\alpha+\gamma, & \beta+2-\zeta \\ \beta, & \beta+3-\zeta\end{matrix}; -\frac{i}{\rho}\right) = \beta\Bigg((\beta+1)E\left(\begin{matrix}\alpha+\gamma, & \beta+2-\zeta \\ \beta+2, & \beta+3-\zeta\end{matrix}; -\frac{i}{\rho}\right)$$

$$-i\rho E\begin{pmatrix}\alpha+\gamma+1, & \beta+3-\zeta \\ \beta+3, & \beta+4-\zeta\end{pmatrix};-\frac{i}{\rho}\Bigg)\Bigg)-i\rho\Bigg((\beta+2)E\begin{pmatrix}\alpha+\gamma+1, & \beta+3-\zeta \\ \beta+3, & \beta+4-\zeta\end{pmatrix};-\frac{i}{\rho}\Bigg)$$
$$-i\rho E\begin{pmatrix}\alpha+\gamma+2, & \beta+4-\zeta \\ \beta+4, & \beta+5-\zeta\end{pmatrix};-\frac{i}{\rho}\Bigg)\Bigg).$$

Substitute the right-hand side of obtained expression in (24b) and get by (3b),

$$\int_0^\rho t^{\beta+1}E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt=\rho^{\beta+2}\Bigg(\beta(\beta+1)\begin{pmatrix}\alpha-1, & \gamma+1 \\ \beta+2, & \beta\end{pmatrix};\frac{i}{\rho}\Bigg)$$
$$-2i\rho(\beta+1)\begin{pmatrix}\alpha-1, & \gamma+2 \\ \beta+3, & \beta\end{pmatrix};\frac{i}{\rho}\Bigg)-\rho^2\begin{pmatrix}\alpha-1, & \gamma+3 \\ \beta+4, & \beta\end{pmatrix};\frac{i}{\rho}\Bigg)\Bigg).$$

For arbitrary $l=0,1,\ldots$, we find by (25),

$$E\begin{pmatrix}\alpha+\gamma, & \beta+l+1-\zeta \\ \beta, & \beta+l+2-\zeta\end{pmatrix};-\frac{i}{\rho}\Bigg)=\sum_{n=0}^{l+1}\frac{(-i\rho)^n\Gamma(\beta+l+1)}{\Gamma(\beta+n)}\binom{l+1}{n}$$
$$\times E\begin{pmatrix}\alpha+\gamma+n, & \beta+l+n+1-\zeta \\ \beta+l+n+1, & \beta+l+n+2-\zeta\end{pmatrix};-\frac{i}{\rho}\Bigg). \tag{27}$$

By substituting the right-hand side of (27) in (24b) and making use of (3b) we finally derive the expression

$$\int_0^\rho t^{\beta+l}E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt$$
$$=\rho^{\beta+l+1}\sum_{n=0}^{l+1}\frac{(-i\rho)^n\Gamma(\beta+l+1)}{\Gamma(\beta+n)}\binom{l+1}{n}\begin{pmatrix}\alpha-1, & \gamma+n+1 \\ \beta+l+n+1, & \beta\end{pmatrix};\frac{i}{\rho}\Bigg) \tag{28}$$

($\operatorname{Re}\beta+l>-1; l=0,1,\ldots$). We note that, if on the left-hand side of (28) we replace (i/t) and $(-i/t)$ accordingly by (λ/t) and $(-\lambda/t)$ for $\lambda\in\mathbb{C}$, then on the right-hand side one needs to replace $(-i\rho)$ in the numerator and (i/ρ) in f by (ρ/λ) and (λ/ρ), respectively. We shall omit the proof since it is obvious. [Hint: starting from (23), one needs to replace $(i\rho)$ and $(-i\rho)$ by $(-\rho/\lambda)$ and (ρ/λ), and then proceed similarly as in the case for $\lambda=i$.]

In [SH03] the authors have shown that [in [SH03, Lemma 1] substitute $a=\alpha$, $b=b'=\beta$, $a'=\alpha+\gamma$, $d=\beta+1$, $k=-k'=i$ and make use of (2a) of the present letter]

$$\int_0^\infty t^\beta e^{-ht} E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt = \frac{\beta\Gamma(\alpha)\Gamma(\alpha+\gamma)}{h^{\beta+1}\Gamma(\beta)}$$
$$\times F_2(\beta+1;\alpha,\alpha+\gamma;\beta,\beta;i/h,-i/h) \qquad (\operatorname{Re}\beta > -1; |h| > 2) \tag{29}$$

where $F_2(\cdot)$ denotes the Appell's hypergeometric function. We shall demonstrate that (29) is easy to obtain from (28).

Recalling that $\exp(-ht) = \sum_{l=0}^\infty (-ht)^l/l!$, we can multiply (28) by $(-h)^l/l!$ and then perform the summation over $l = 0,1,\ldots$ This gives the integral

$$\int_0^\rho t^\beta e^{-ht} E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt = \rho^{\beta+1}\sum_{l=0}^\infty \frac{(-\rho h)^l}{l!}$$
$$\times \sum_{n=0}^{l+1} \frac{(-i\rho)^n\Gamma(\beta+l+1)}{\Gamma(\beta+n)}\binom{l+1}{n}\left(\begin{matrix}\alpha-1, & \gamma+n+1\\ \beta+l+n+1, & \beta\end{matrix};\frac{i}{\rho}\right) \tag{30}$$

($\operatorname{Re}\beta > -1$). Applying (3b) to the function f on the right-hand side of (30), one finds by virtue of (27),

$$\int_0^\rho t^\beta e^{-ht} E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt = \frac{\rho^{\beta+1}}{2\pi i}$$
$$\times \int_B d\zeta \frac{(i/\rho)^\zeta\Gamma(\zeta)\Gamma(\alpha-\zeta)}{\Gamma(\beta-\zeta)}\sum_{l=0}^\infty \frac{(-\rho h)^l}{l!} E\left(\begin{matrix}\alpha+\gamma, & \beta+l+1-\zeta\\ \beta, & \beta+l+2-\zeta\end{matrix};-\frac{i}{\rho}\right).$$

By applying (2b) to E-function,

$$\int_0^\rho t^\beta e^{-ht} E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt = \frac{\rho^{\beta+1}}{2\pi i}$$
$$\times \int_B d\zeta \frac{(i/\rho)^\zeta\Gamma(\zeta)\Gamma(\alpha-\zeta)}{\Gamma(\beta-\zeta)}\cdot\frac{1}{2\pi i}\int_B d\zeta' \frac{(-i/\rho)^{\zeta'}\Gamma(\zeta')\Gamma(\alpha+\gamma-\zeta')}{\Gamma(\beta-\zeta')}$$
$$\times \sum_{l=0}^\infty \frac{(-\rho h)^l}{l!}\cdot\frac{1}{\beta+l+1-\zeta-\zeta'}. \tag{31}$$

The sum over l equals $(\rho h)^{-\beta-1+\zeta+\zeta'}(\Gamma(\beta+1-\zeta-\zeta')-\Gamma(\beta+1-\zeta-\zeta',\rho h))$, where the incomplete gamma function $\Gamma(a,t) = \int_t^\infty u^{a-1}e^{-u}du$.

Starting from now, we can either (i) take the limit $\rho\to\infty$ so that $\Gamma(\cdot,\rho h)\to 0$ or (ii) apply the above given integral representation of the incomplete gamma function for arbitrary $\rho\in\mathbb{R}$. Both cases yield identical results.

In case (i), the integral (31) becomes equal to

$$\int_0^\infty t^\beta e^{-ht} E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt = \frac{h^{-\beta-1}}{2\pi i}\int_B d\zeta \frac{(ih)^\zeta\Gamma(\zeta)\Gamma(\alpha-\zeta)}{\Gamma(\beta-\zeta)}$$
$$\times \frac{1}{2\pi i}\int_B d\zeta' \frac{(-ih)^{\zeta'}\Gamma(\zeta')\Gamma(\alpha+\gamma-\zeta')\Gamma(\beta+1-\zeta-\zeta')}{\Gamma(\beta-\zeta')}.$$

The latter contour integral is calculated by applying the residue theorem,

$$\frac{1}{2\pi i}\int_B d\zeta' \frac{(-ih)^{\zeta'}\Gamma(\zeta')\Gamma(\alpha+\gamma-\zeta')\Gamma(\beta+1-\zeta-\zeta')}{\Gamma(\beta-\zeta')}$$
$$= \sum_{\nu=0}^{\infty} \frac{(-i/h)^\nu\Gamma(\alpha+\gamma+\nu)\Gamma(\beta+1-\zeta+\nu)}{\nu!\Gamma(\beta+\nu)} = \frac{\Gamma(\alpha+\gamma)\Gamma(\beta+1-\zeta)}{\Gamma(\beta)}$$
$$\times {}_2F_1(\alpha+\gamma,\beta+1-\zeta;\beta;-i/h),$$

and thus

$$\int_0^\infty t^\beta e^{-ht} E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt = \frac{\Gamma(\alpha+\gamma)}{h^{\beta+1}\Gamma(\beta)}\cdot\frac{1}{2\pi i}\int_B d\zeta$$
$$\times (ih)^\zeta\Gamma(\zeta)\Gamma(\alpha-\zeta)(\beta-\zeta)\,{}_2F_1(\alpha+\gamma,\beta+1-\zeta;\beta;-i/h)$$

or by the residue theorem,

$$\int_0^\infty t^\beta e^{-ht} E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt$$
$$= \frac{\Gamma(\alpha+\gamma)}{h^{\beta+1}\Gamma(\beta)}\sum_{\nu=0}^{\infty}\frac{(i/h)^\nu}{\nu!}\Gamma(\alpha+\nu)(\beta+\nu)\,{}_2F_1(\alpha+\gamma,\beta+1+\nu;\beta;-i/h)$$
$$= \frac{\beta\Gamma(\alpha)\Gamma(\alpha+\gamma)}{h^{\beta+1}\Gamma(\beta)}\sum_{\nu=0}^{\infty}\frac{(i/h)^\nu(\alpha)_\nu(\beta+1)_\nu}{\nu!(\beta)_\nu}\,{}_2F_1(\alpha+\gamma,\beta+1+\nu;\beta;-i/h).$$

The latter sum is nothing more than the Appell's double series given on the right-hand side of (29). The series $F_2(\cdot)$ is absolutely convergent for $|h|>2$. Hence (29) holds.

In case (ii), we obtain from (31), by applying the residue theorem as above,

$$\int_0^\rho t^\beta e^{-ht} E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt = \frac{\beta\Gamma(\alpha)\Gamma(\alpha+\gamma)}{h^{\beta+1}\Gamma(\beta)}$$
$$\times F_2(\beta+1;\alpha,\alpha+\gamma;\beta,\beta;i/h,-i/h) - \frac{h^{-\beta-1}}{2\pi i}\int_B d\zeta \frac{(ih)^\zeta\Gamma(\zeta)\Gamma(\alpha-\zeta)}{\Gamma(\beta-\zeta)}$$

$$\times \frac{1}{2\pi i}\int_B d\zeta' \frac{(-ih)^{\zeta'}\Gamma(\zeta')\Gamma(\alpha+\gamma-\zeta')\Gamma(\beta+1-\zeta-\zeta',\rho h)}{\Gamma(\beta-\zeta')}, \tag{32}$$

where the contour integral involving the incomplete gamma function is calculated by using the integral representation of $\Gamma(\cdot,\rho h)$ along with the residue theorem, namely,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_B d\zeta' \frac{(-ih)^{\zeta'}\Gamma(\zeta')\Gamma(\alpha+\gamma-\zeta')\Gamma(\beta+1-\zeta-\zeta',\rho h)}{\Gamma(\beta-\zeta')}\\
&=\sum_{\nu=0}^{\infty}\frac{(-i/h)^{\nu}\Gamma(\alpha+\gamma+\nu)\Gamma(\beta+1-\zeta+\nu,\rho h)}{\nu!\Gamma(\beta+\nu)}=\int_{\rho h}^{\infty} dt\, t^{\beta-\zeta}e^{-t}\\
&\times\sum_{\nu=0}^{\infty}\frac{(-it/h)^{\nu}\Gamma(\alpha+\gamma+\nu)}{\nu!\Gamma(\beta+\nu)}=\int_{\rho h}^{\infty} t^{\beta-\zeta}e^{-t}E(\alpha+\gamma;\beta;-ih/t)dt\\
&=h^{\beta+1-\zeta}\int_{\rho}^{\infty} t^{\beta-\zeta}e^{-ht}E(\alpha+\gamma;\beta;-i/t)dt,
\end{aligned} \tag{33}$$

where in the last step the substitution $t\to ht$ has been initiated.

Substitute (33) in (32) and get by (2b),

$$\begin{aligned}
&\int_0^{\rho} t^{\beta}e^{-ht}E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt=\frac{\beta\Gamma(\alpha)\Gamma(\alpha+\gamma)}{h^{\beta+1}\Gamma(\beta)}\\
&\times F_2(\beta+1;\alpha,\alpha+\gamma;\beta,\beta;i/h,-i/h)\\
&-\int_{\rho}^{\infty} t^{\beta}e^{-ht}E(\alpha;\beta;i/t)E(\alpha+\gamma;\beta;-i/t)dt.
\end{aligned}$$

But $\int_0^{\rho}+\int_{\rho}^{\infty}=\int_0^{\infty}$ for all $\rho\in\mathbb{R}$, and hence (29) holds again.

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